Section 3.3 Concavity and the Second Derivative Test
(Minimum Homework: 1 – 24 odds)

We will study the concept of **concavity of a graph** in this section.

Here is a less than stellar explanation of the concept of concavity of a region of the graph of a function.

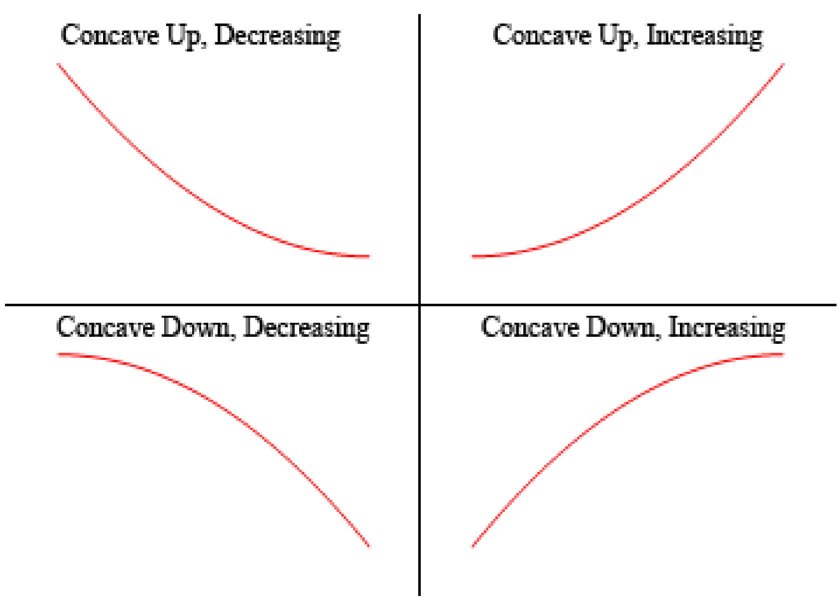
If a region in the graph of a function is not a straight line, we say the graph is either concave up in the region or concave down in the region.

The graph of a function is **concave up** in a region if it "opens" up
The graph of a function is **concave down** in a region if it "opens" down.

Here is an image that might make clearer what I am trying to say.

Notice that concavity has nothing to do with increasing or decreasing.

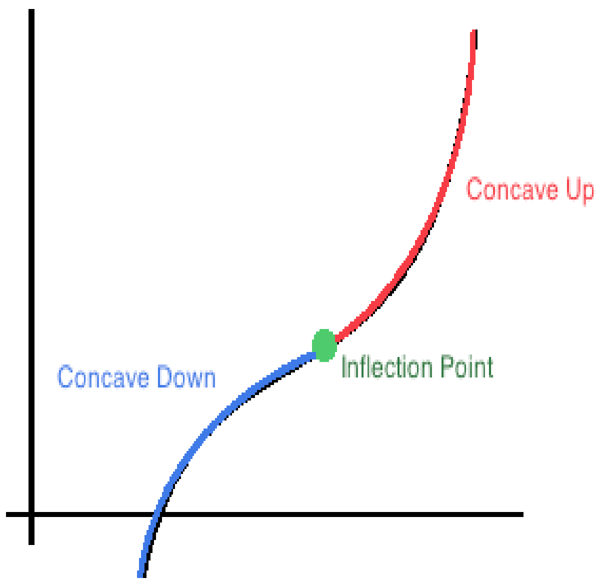
- A function can be concave up and either increasing or decreasing.
- A function can be concave down and either increasing or decreasing.

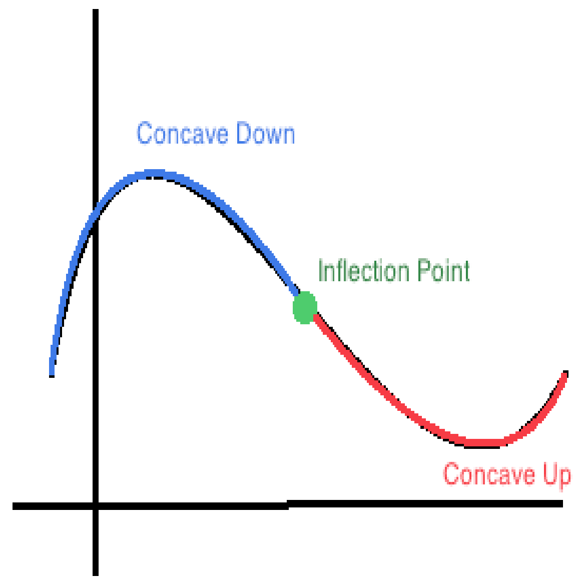


A point on a graph is called an **inflection point,** provided:

- The graph is continuous at the point.
- The graph changes concavity at the point.

Here are graphs that illustrate the concept of an inflection point.





Intervals where the graph of a function is concave up / concave down are found much like intervals where a graph is increasing and decreasing. It is a little trickier to decide on the correct intervals.

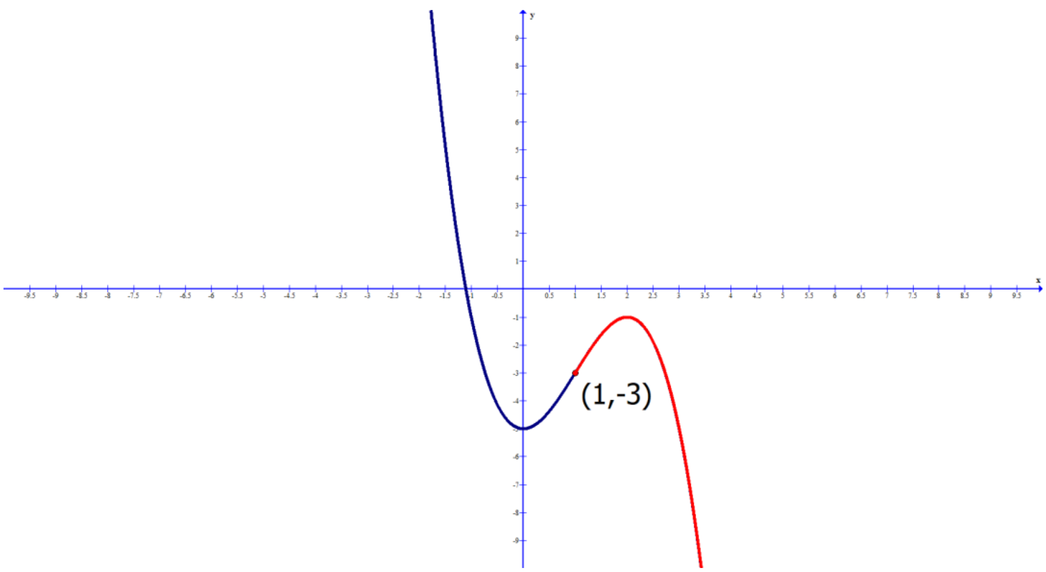
How to determine the open intervals where a graph is concave up / concave down from a graph of a function.

- Find the x-coordinate of all inflection points.
- Find the x-coordinate of any hole / vertical asymptote.

Place numbers on a number line that also contains: $-\infty \; and \; \infty$

Create intervals that only have round parenthesis.

Determine if the graph is concave up or concave down in the interval by inspection.



The x-coordinate of the inflection point is $x = 1$
There is no hole / vertical asymptote.

Create a number line that contains $-\infty, \infty$ and 1 and create intervals with round parenthesis. Determine concavity by inspection.

	$(-\infty, 1)$	$(1, \infty)$
$-\infty$	1	∞
	Concave up by inspection	Concave down by inspection

$Answer$: $Concave\ up\ \ (-\infty, 1)\qquad Concave\ down\ \ (1, \infty)$

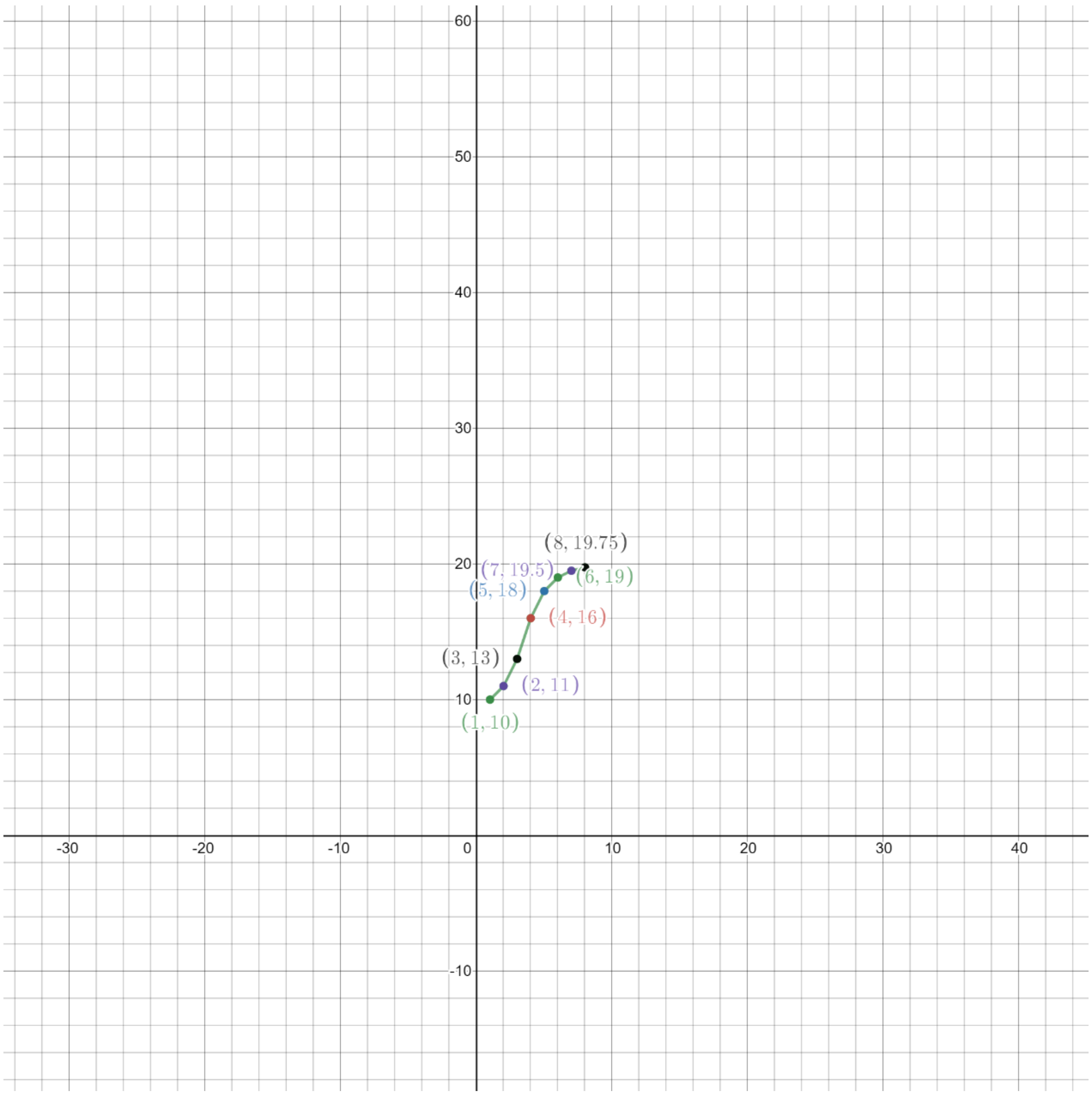
This example may help you understand concavity.

A company produces a component for cell phones.   Due to enormous demand the company has been able to raise the price of the product every year for the last 8 years. The table shows the price the company charged for the component for each of the last 8 years.

Year	1	2	3	4	5	6	7	8
Price (in dollars)	10	11	13	16	18	19	19.5	19.75

The price of the component is increasing for the entire 8 years.

Here is a plot of the data in form (year, price) I connected the points to help you see that the inflection point is the point $(4,16)$



60
50
40
30
20
10
-10
-30
-20
-10
0
10
20
30
40
(8, 19.75)
(7, 19.5)
(6, 19)
(5, 18)
(4, 16)
(3, 13)
(2, 11)
(1, 10)

Let me add a yearly price increase row to the table.

Year	1	2	3	4	5	6	7	8
Price (in dollars)	10	11	13	16	18	19	19.5	19.75
Yearly Price increase	\$1	\$2	\$3	\$2	\$1	\$0.50	\$0.75	

The graph is increasing over the entire 8 years it represents.

The graph is concave up from year one to year 4.

- Because the price increase is greater in each successive year

The graph is concave down from year 4 to year 8.

- Because the price increase is less in each successive year

$(4{,}16)$ is an inflection point.

- Year 4 is when the price changes start going up by smaller amounts.

Now we need to use Calculus to find:

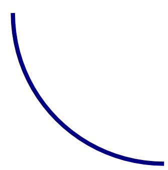
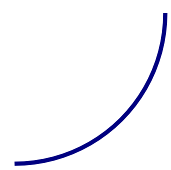
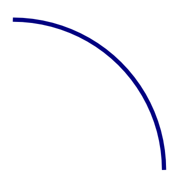
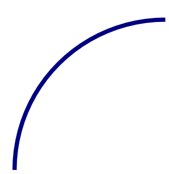
- Intervals where the graph of a function is Concave Up
- Intervals where the graph of a function is Concave Down
- All inflection points

The graph of function is **Concave Up** in an interval when the value of the **second derivative** for each value of x in the interval is **positive**.

The graph of a function is **Concave Down** in an interval when the value of the **second derivative** for each value of x in the interval is **negative.**

This table summarizes the tests for increasing and decreasing intervals, and for intervals that are concave up and concave down.

First vertical column: Graph is decreasing when 1st derivative is negative.
Second vertical column: Graph is increasing when 1st derivative is positive.
First horizontal row: Graph is concave up when second derivative is positive.
Second horizontal row: Graph is concave down when second derivative is negative.

	$f'(x) < 0$	$0 < f'(x)$
$0 < f''(x)$	 Here $y = f(x)$ is decreasing, while the rate itself is increasing. In this case the curve is **concave up**.	 Here $y = f(x)$ is increasing, while the rate itself is increasing. In this case the curve is **concave up**.
$f''(x) < 0$	 Here $y = f(x)$ is decreasing, while the rate itself is decreasing. In this case the curve is **concave down**.	 Here $y = f(x)$ is increasing, while the rate itself is decreasing. In this case the curve is **concave down**.

Steps to determine:

- Intervals where the graph of a function is Concave Up
- Intervals where the graph of a function is Concave Down
- Inflection points

1) Find the second derivative of the function.
2) Find all critical numbers for the second derivative.

- Any value that makes the second derivative equal to zero
- Any value that makes the second derivative undefined

3) Plot the critical numbers on a number line that also includes $-\infty\ and\ \infty$.
4) Create interval(s) using only round parenthesis.
5) Pick a number inside the interval and plug it into the derivative.
6) Determine whether the graph is concave up or concave down in that interval.

- Graph is concave up when the result is positive.
- Graph is concave down when the result is negative.

7) Determine if the critical point is an inflection point. A critical number is an inflection point when:

- The graph is continuous at the point.
- The graph changes concavity at the point.

8) Find the y-coordinates of any inflection points.

9) Write your answer.

Example: $f(x) = 2x^3 - 12x^2 + 5$

a) Find the open interval(s) where the function is concave up

b) Find the open interval(s) where the graph of the function is concave down.

c) Find all inflection points

Steps:

1) Find the second derivative of the function.

$f'(x) = 6x^2 - 24x$

$f''(x) = 12x - 24$

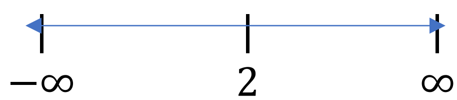
2) Find all critical numbers for the second derivative.

$12x - 24 = 0$

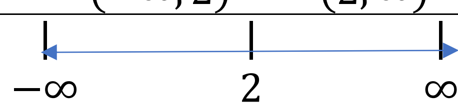
$12x = 24$

Critical number $x = 2$

3) Plot the critical number(s) on a number line that also includes $-\infty$ *and* ∞.



4) Create interval(s) using only round parenthesis.

$(-\infty, 2)$ $(2, \infty)$


5) Pick a number inside the interval and plug it into the derivative.

Interval: $(-\infty, 2)$
Check $x = 0$
$f''(0) = 12(0) - 24 \; = \; -24$

Interval: $(2, \infty)$
Check $x = 3$
$f''(3) = 12(3) - 24 \; = \; 12$

6) Determine whether the graph is concave up or concave down in that interval.
Interval: $(-\infty, 2)$ $concave\ down\ since\ \ f''\ is\ negative$

Interval $(2, \infty)$ $concave\ up\ since\ f''\ is\ positive$

7) Determine if the critical point is an inflection point. A critical number is an inflection point when:

The graph changes concavity at $x = 2$ and the graph is continuous at $x = 2$.
$x = 2$ is the x-coordinate of an inflection point.

8) Find the y-coordinates of any inflection points.

$y = f(2) = 2(2)^3 - 12(2)^2 + 5 = -27$

Inflection point $(2, -27)$

9) Write your answer.

a) Find the open interval(s) where the function is concave up $(2, \infty)$

b) Find the open interval(s) where the graph of the function is concave down. $(-\infty, 2)$

c) Find all inflection points $(2, -27)$

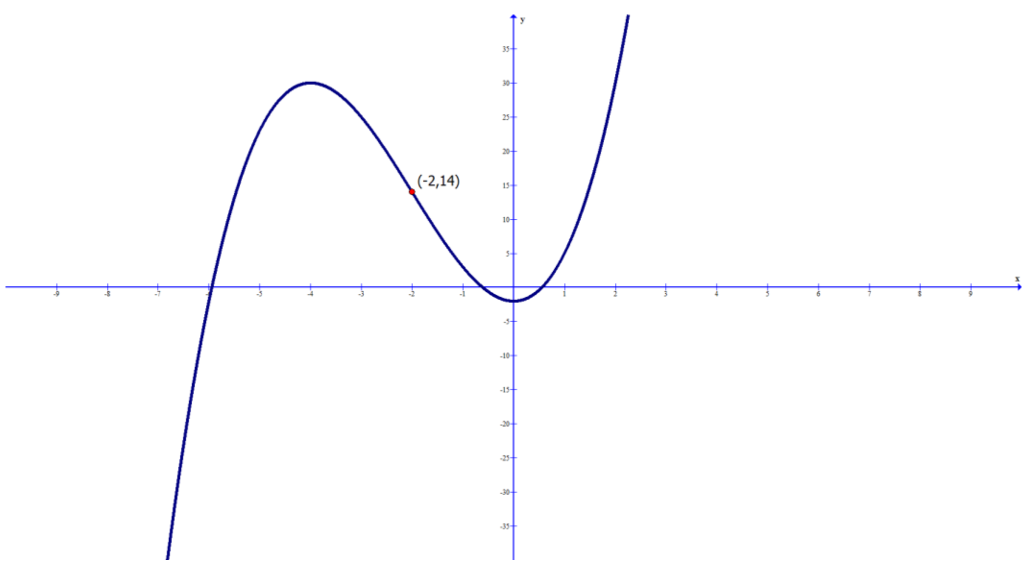
#1-14:

a) Find the open interval(s) where the graph of the function is concave up

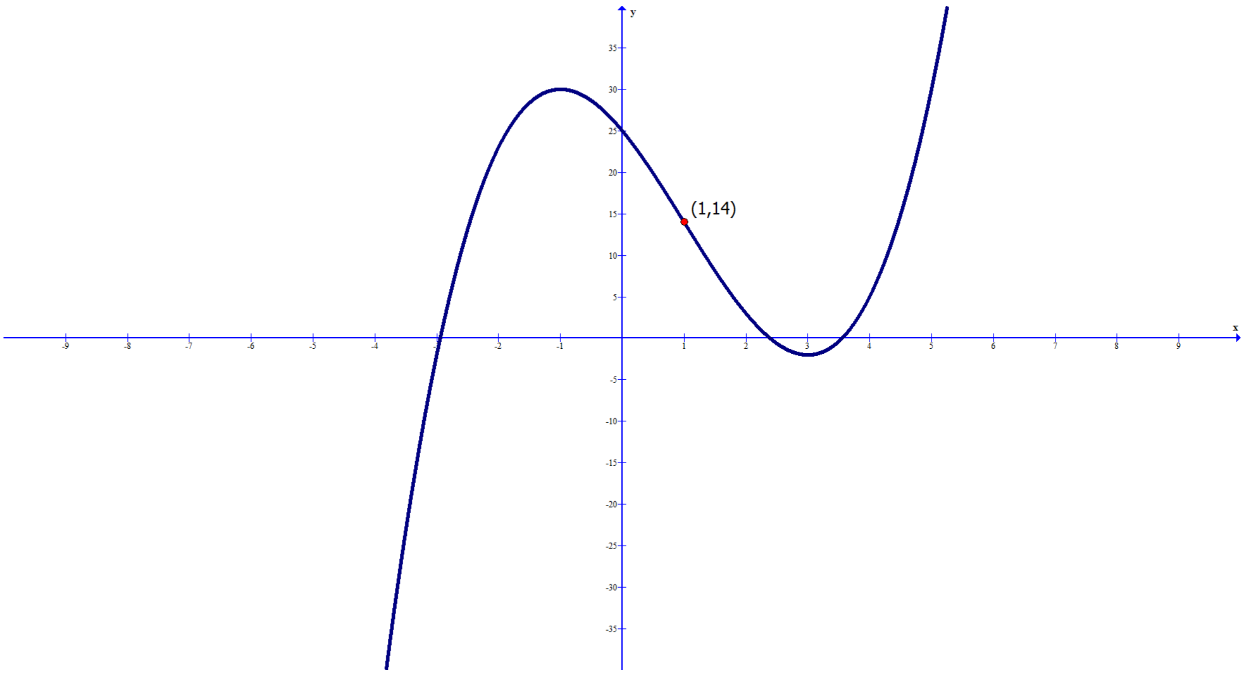
b) Find the open interval(s) where the graph of the function is concave down.

c) Find all inflection points

1)



2)

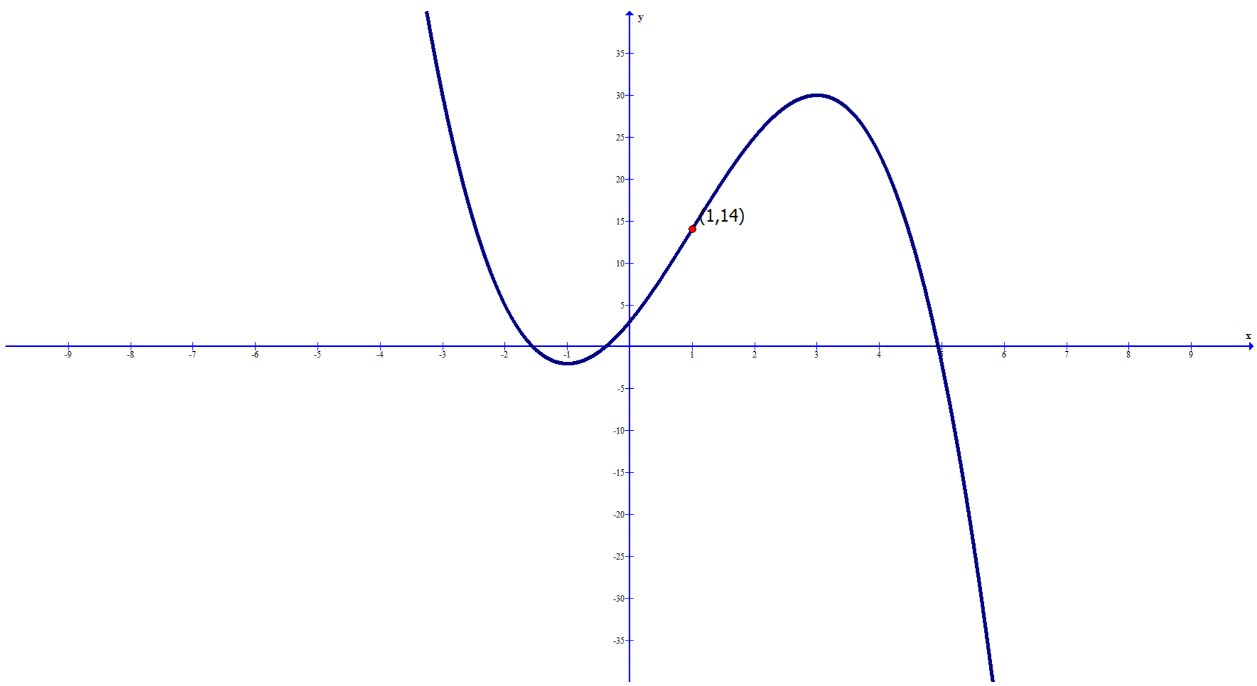


2a) Find the open interval(s) where the graph of the function is concave up $(1, \infty)$

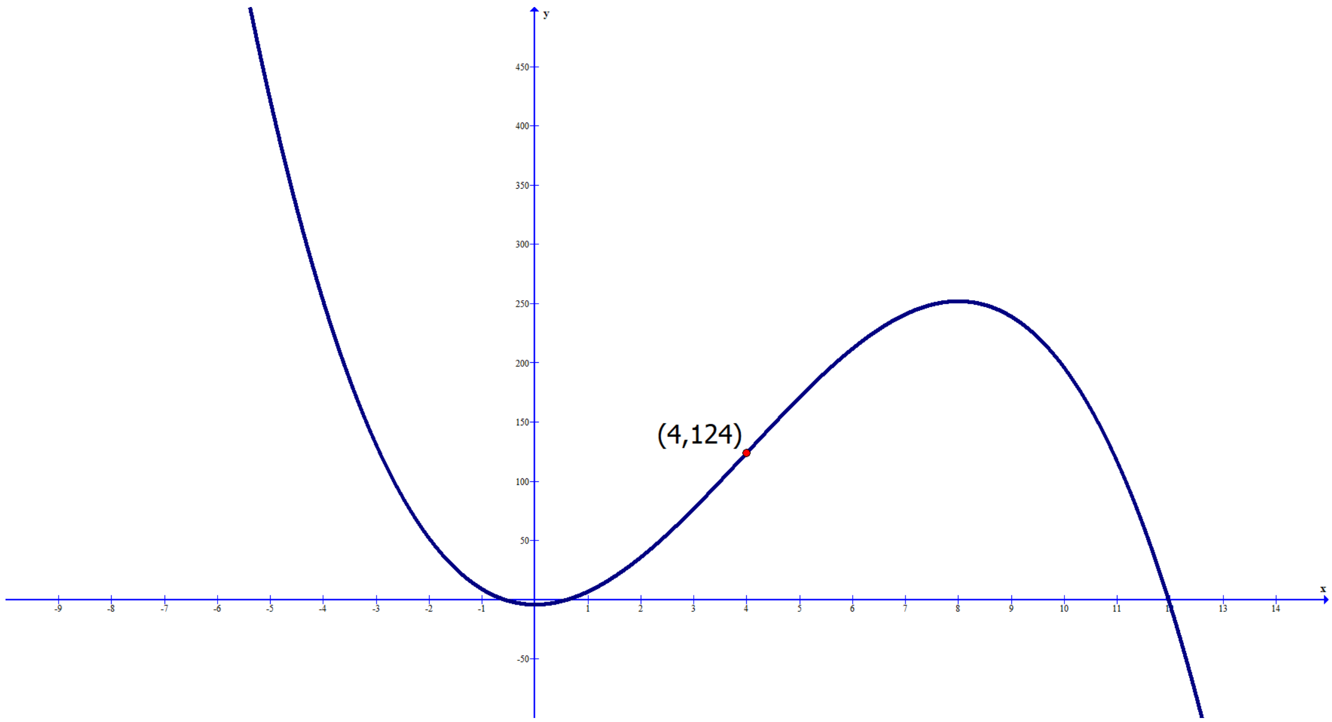
2b) Find the open interval(s) where the graph of the function is concave down. $(-\infty, 1)$

2c) Find all inflection points $(1,14)$

3)



4)

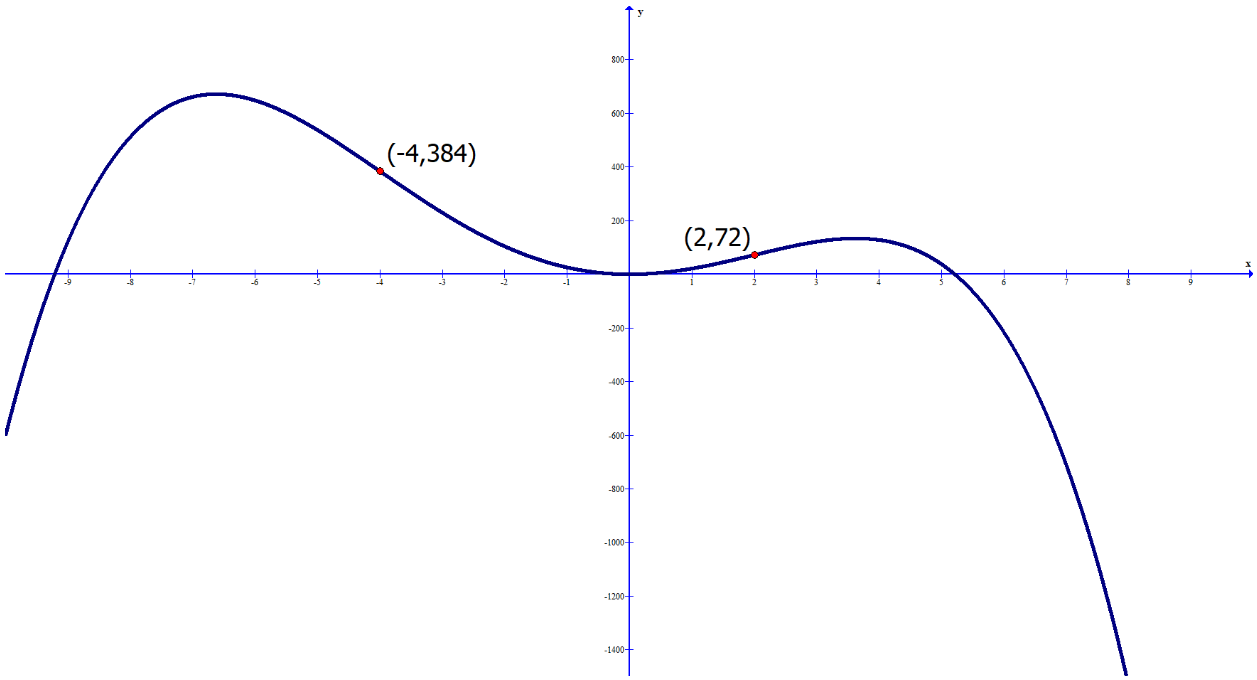


4a) Find the open interval(s) where the graph of the function is concave up $(-\infty, 4)$

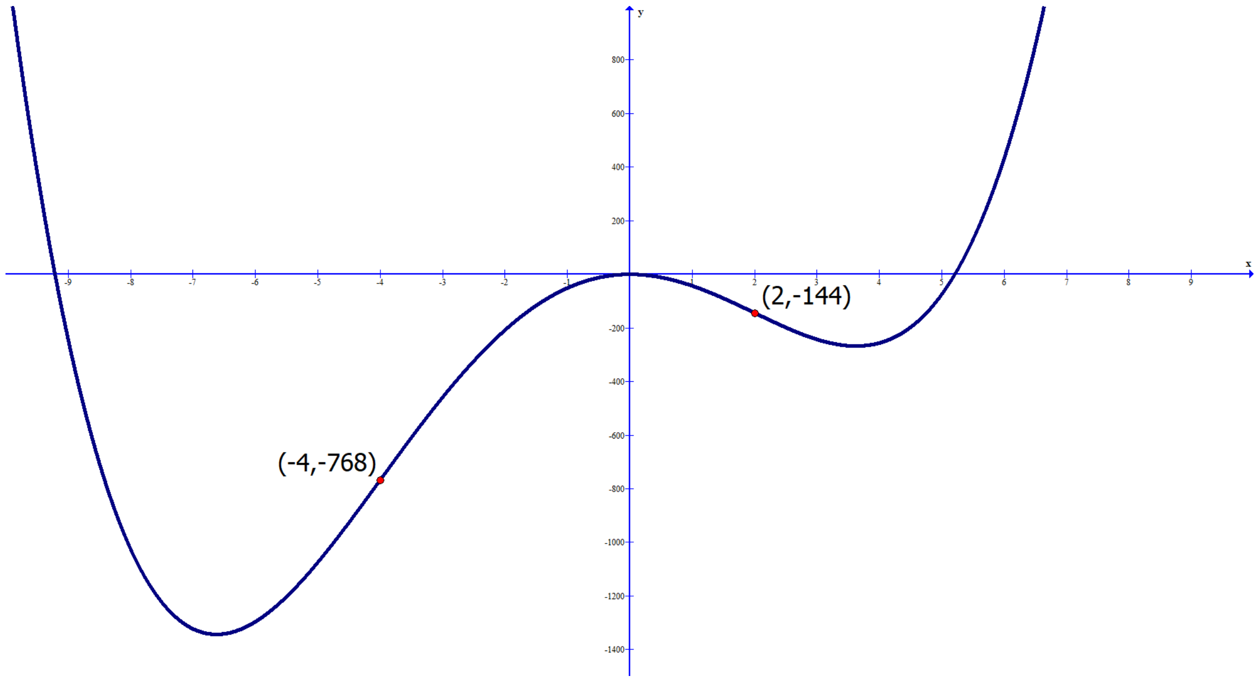
4b) Find the open interval(s) where the graph of the function is concave down. $(4, \infty)$

4c) Find all inflection points $(4{,}124)$

5)


y
x
(-4,384)
(2,72)
800
600
400
200
-200
-400
-600
-800
-1000
-1200
-1400
-9
-8
-7
-6
-5
-4
-3
-2
-1
1
2
3
4
5
6
7
8
9

6)

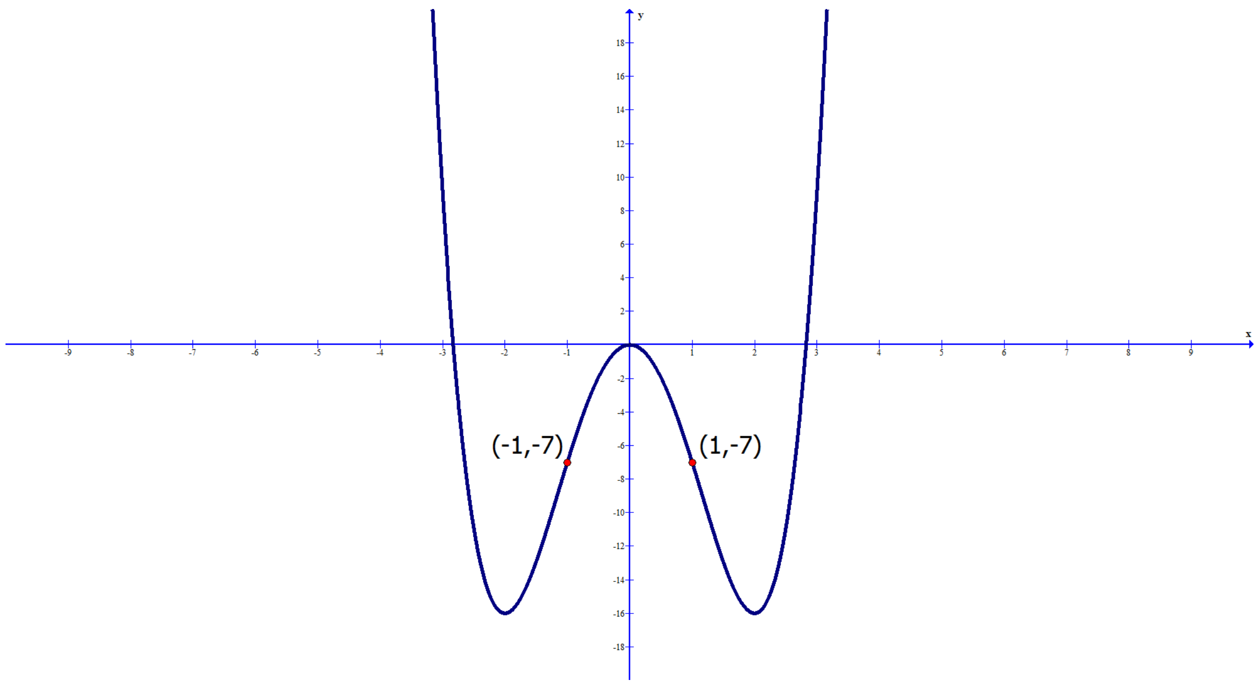


6a) Find the open interval(s) where the graph of the function is concave up $(-\infty, -4) \cup (2, \infty)$

6b) Find the open interval(s) where the graph of the function is concave down. $(-4,2)$

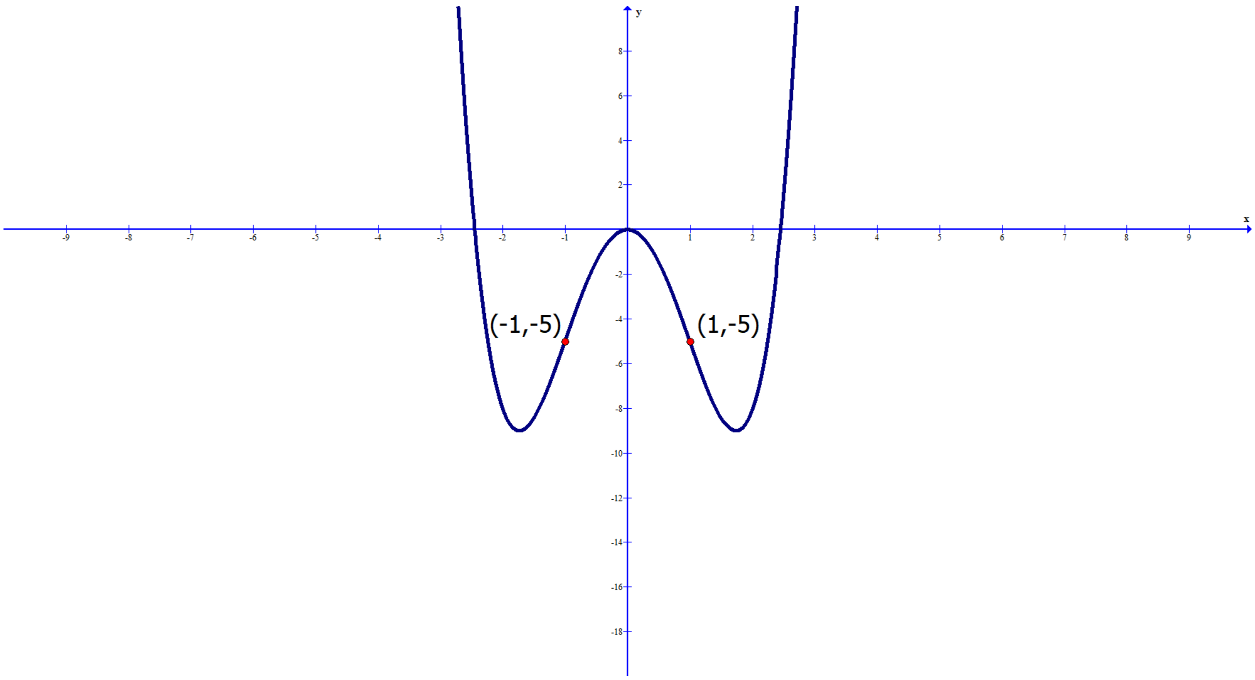
6c) Find all inflection points $(-4, -768)$ *and* $(2, -144)$

7)



y
x
18
16
14
12
10
8
6
4
2
-2
-4
-6
-8
-10
-12
-14
-16
-18
-9
-8
-7
-6
-5
-4
-3
-2
-1
1
2
3
4
5
6
7
8
9
(-1,-7)
(1,-7)

8)

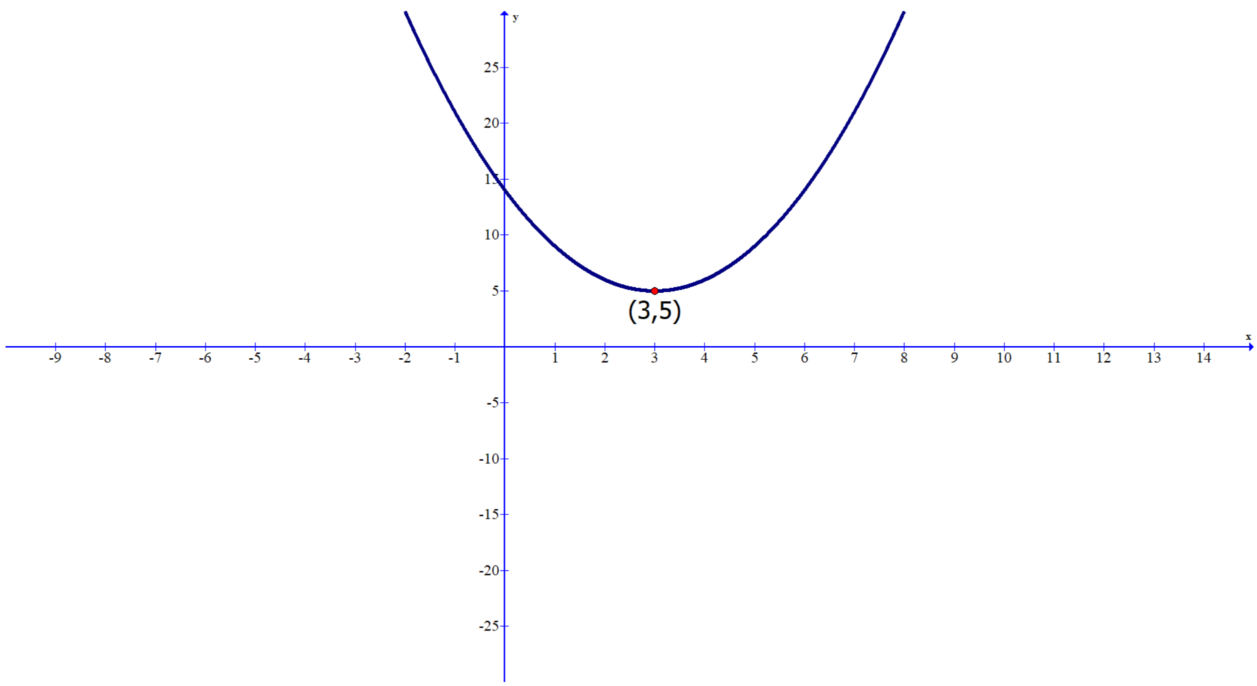


8a) Find the open interval(s) where the graph of the function is concave up $(-\infty,-1)\cup(1,\infty)$

8b) Find the open interval(s) where the graph of the function is concave down. $(-1,1)$

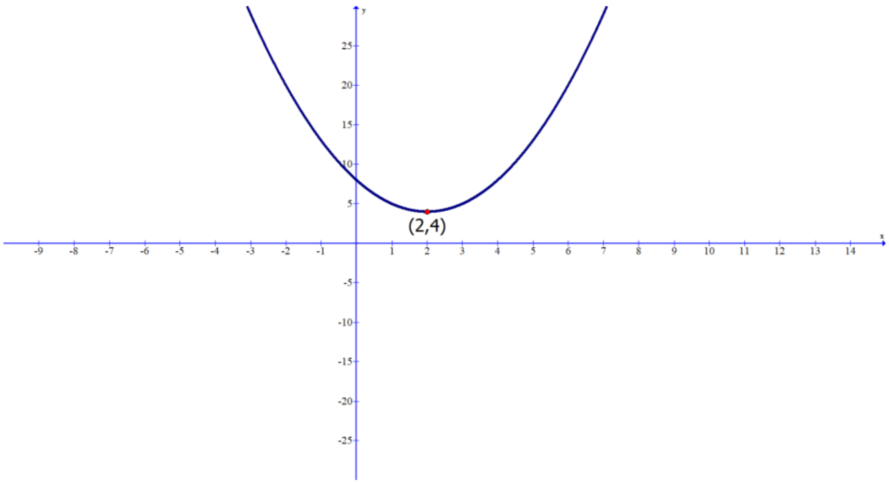
8c) Find all inflection points $(-1,-5)$ *and* $(1,-5)$

9)



y
x
25
20
15
10
5
-5
-10
-15
-20
-25
-9
-8
-7
-6
-5
-4
-3
-2
-1
1
2
3
4
5
6
7
8
9
10
11
12
13
14
(3,5)

10)

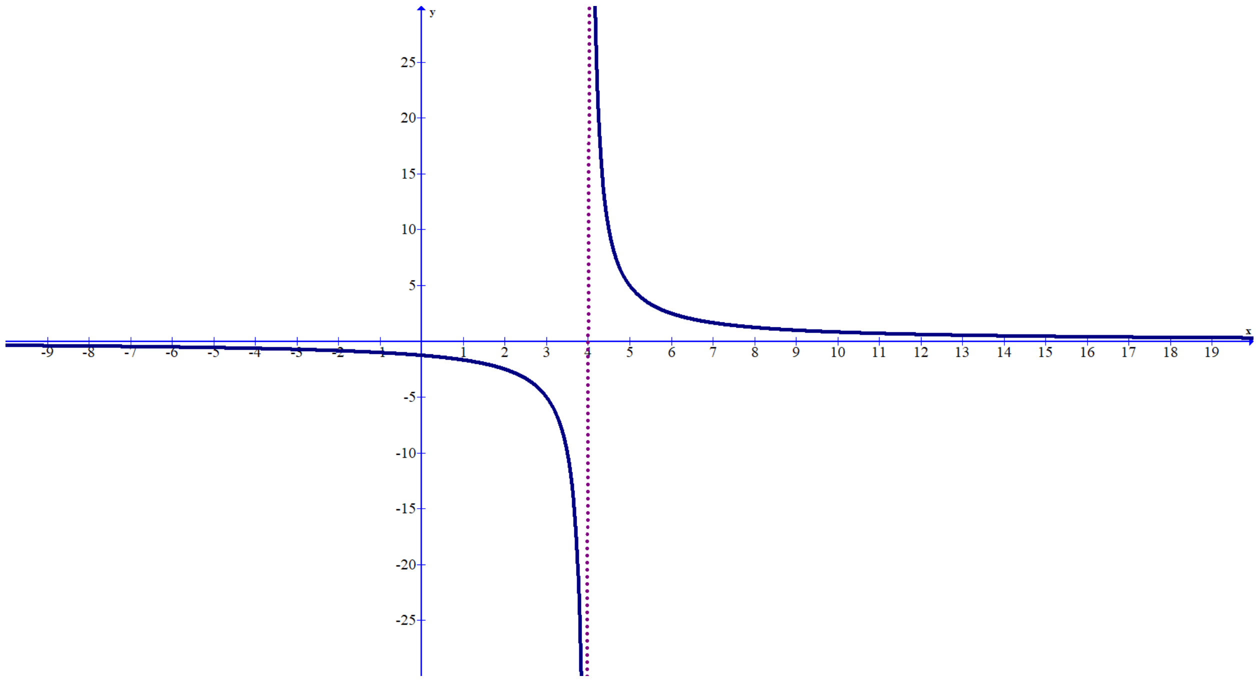


10a) Find the open interval(s) where the graph of the function is concave up $(-\infty, \infty)$

10b) Find the open interval(s) where the graph of the function is concave down. none

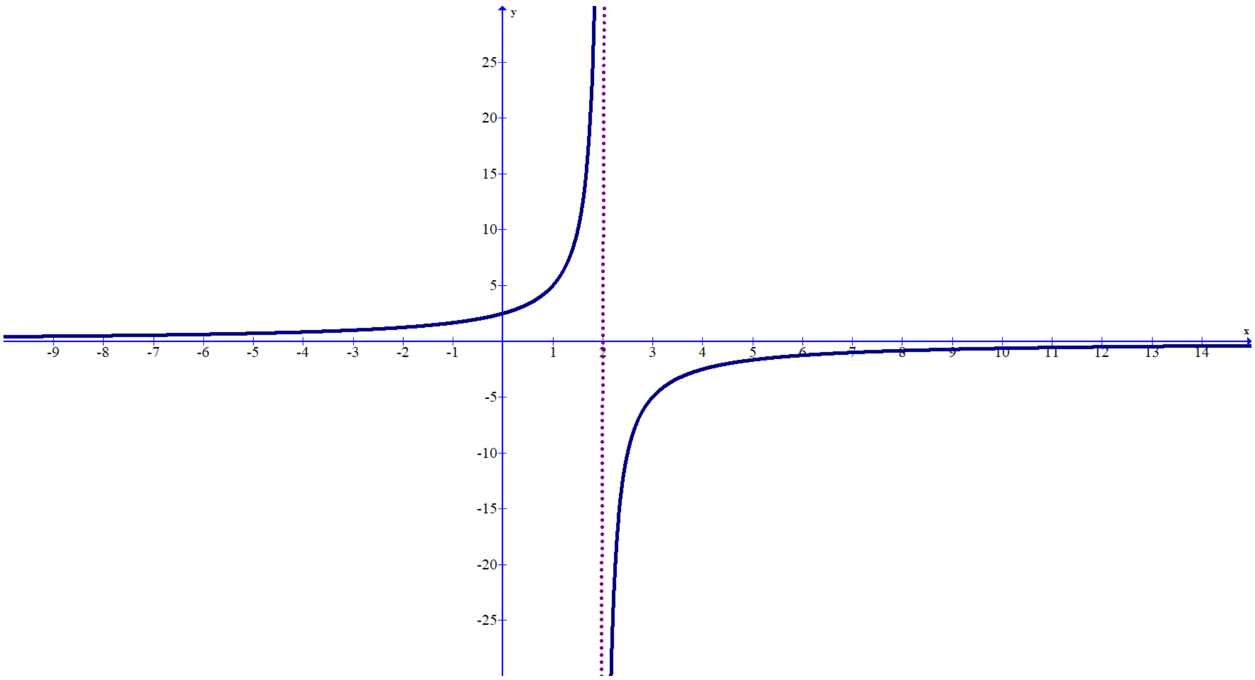
10c) Find all inflection points none

11)



y
x
25
20
15
10
5
-5
-10
-15
-20
-25
-9
-8
-7
-6
-5
-4
-3
-2
-1
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19

12)

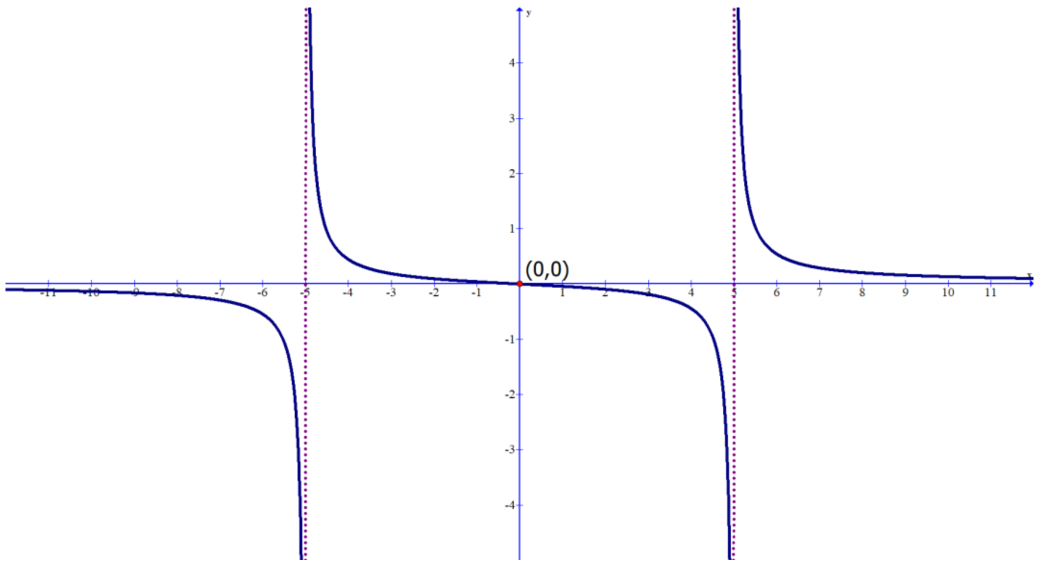


12a) Find the open interval(s) where the graph of the function is concave up $(-\infty, 2)$

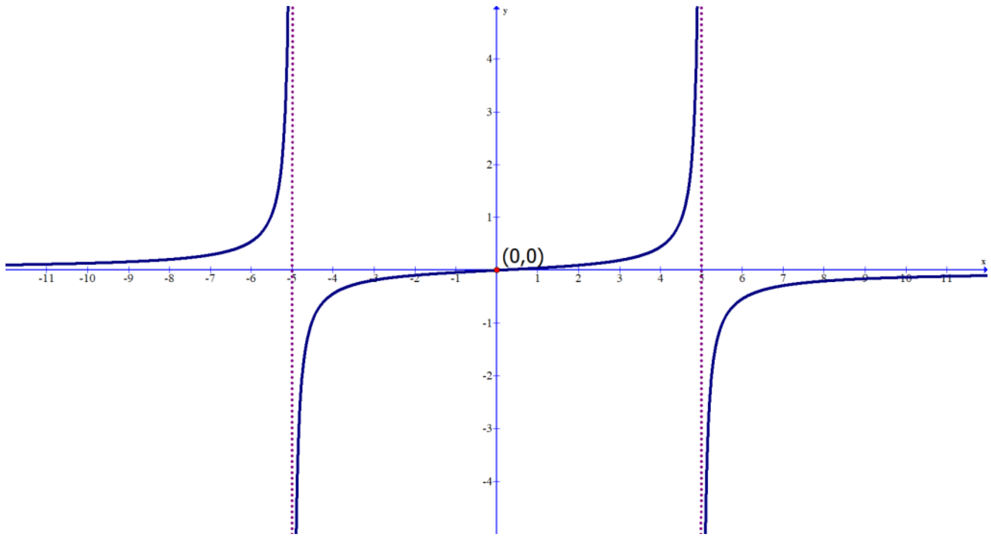
12b) Find the open interval(s) where the graph of the function is concave down. $(2, \infty)$

12c) Find all inflection points $none, as\ x = 2\ is\ not\ in\ the\ domain\ of\ the\ function\ graphed$

13)



14)



14a) Find the open interval(s) where the graph of the function is concave up $(-\infty,-5)\cup(0,5)$

14b) Find the open interval(s) where the graph of the function is concave down. $(-5,0)\cup(5,\infty)$

14c) Find all inflection points $(0,0)$

#15-24:

a) Find the open interval(s) where the graph of the function is concave up

b) Find the open interval(s) where the graph of the function is concave down.

c) Find all inflection points

15) $f(x) = x^3 - 3x^2 + 5$

16) $f(x) = 2x^3 - 6x^2 + 5$

1) Find the second derivative of the function.

2) Find all critical numbers for the second derivative.

3) Plot the critical numbers on a number line that also includes $-\infty\ and\ \infty$.

4) Create interval(s) using only round parenthesis.

5) Pick a number inside the interval and plug it into the derivative.

6) Determine whether the graph is concave up or concave down in that interval.

7) Determine if the critical point is an inflection point. A critical number is an inflection point when:

8) Find the y-coordinates of any inflection points.

9) Write your answer.

16a) Find the open interval(s) where the graph of the function is concave up $(1, \infty)$

16b) Find the open interval(s) where the graph of the function is concave down. $(-\infty, 1)$

16c) Find all inflection points $(1,1)$

17) $f(x) = -x^3 - 3x^2 + 5$

18) $f(x) = -2x^3 - 6x^2 + 5$

1) Find the second derivative of the function.

2) Find all critical numbers for the second derivative.

3) Plot the critical numbers on a number line that also includes $-\infty\ and\ \infty$.

4) Create interval(s) using only round parenthesis.

5) Pick a number inside the interval and plug it into the derivative.

6) Determine whether the graph is concave up or concave down in that interval.

7) Determine if the critical point is an inflection point. A critical number is an inflection point when:

8) Find the y-coordinates of any inflection points.

9) Write your answer.

18a) Find the open interval(s) where the graph of the function is concave up $(-\infty, -1)$

18b) Find the open interval(s) where the graph of the function is concave down. $(-1, \infty)$

18c) Find all inflection points $(-1,1)$

19) $f(x) = x^4 - 6x^2 + 4$

20) $f(x) = x^4 - 6x^2 - 3$

1) Find the second derivative of the function.

2) Find all critical numbers for the second derivative.

3) Plot the critical numbers on a number line that also includes $-\infty\ and\ \infty$.

4) Create interval(s) using only round parenthesis.

5) Pick a number inside the interval and plug it into the derivative.

6) Determine whether the graph is concave up or concave down in that interval.

7) Determine if the critical point is an inflection point. A critical number is an inflection point when:

8) Find the y-coordinates of any inflection points.

9) Write your answer.

20a) Find the open interval(s) where the graph of the function is concave up $(-\infty, -1) \cup (1, \infty)$

20b) Find the open interval(s) where the graph of the function is concave down. $(-1,1)$

20c) Find all inflection points $(-1, -8)$ *and* $(1, -8)$

21) $f(x) = 2xe^x$

22) $f(x) = 3xe^x$

1) Find the second derivative of the function.

2) Find all critical numbers for the second derivative.

3) Plot the critical numbers on a number line that also includes $-\infty \; and \; \infty$.

4) Create interval(s) using only round parenthesis.

5) Pick a number inside the interval and plug it into the derivative.

6) Determine whether the graph is concave up or concave down in that interval.

7) Determine if the critical point is an inflection point. A critical number is an inflection point when:

8) Find the y-coordinates of any inflection points.

9) Write your answer.

22a) Find the open interval(s) where the graph of the function is concave up $(-2, \infty)$

22b) Find the open interval(s) where the graph of the function is concave down. $(-\infty, -2)$

22c) Find all inflection points $\left(-2, \frac{-6}{e^2}\right)$

23) f(x) = $\frac{2}{x-5}$

Hint $f''(x) = \frac{4}{(x-5)^3}$

24) f(x) = $\frac{5}{x+1}$

Hint: $f"(x) = \frac{10}{(x+1)^3}$

1) Find the second derivative of the function.

2) Find all critical numbers for the second derivative.

3) Plot the critical numbers on a number line that also includes $-\infty \ and \ \infty$.

4) Create interval(s) using only round parenthesis.

5) Pick a number inside the interval and plug it into the derivative.

6) Determine whether the graph is concave up or concave down in that interval.

7) Determine if the critical point is an inflection point. A critical number is an inflection point when:

8) Find the y-coordinates of any inflection points.

9) Write your answer.

24a) Find the open interval(s) where the graph of the function is concave up $(-1, \infty)$

24b) Find the open interval(s) where the graph of the function is concave down. $(-\infty, -1)$

24c) Find all inflection points $none, as\ x = -1\ is\ not\ in\ the\ domain\ of\ the\ function$